

## Self-organization to criticality in a system without conservation law

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# Self-organization to criticality in a system without conservation law

Stefano Lise

Department of Mathematics, Huxley Building, Imperial College of Science, Technology, and Medicine, London SW7 2BZ, UK

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## Abstract

We numerically investigate the approach to the stationary state in the non-conservative Olami–Feder–Christensen (OFC) model for earthquakes. Starting from initially random configurations, we monitor the average earthquake size in different portions of the system as a function of time (the time is defined as the input energy *per site* in the system). We find that the process of self-organization develops from the boundaries of the system and it is controlled by a dynamical critical exponent  $z \simeq 1.3$  that appears to be universal over a range of dissipation levels of the local dynamics. We show moreover that the transient time of the system  $t_{tr}$  scales with system size  $L$  as  $t_{tr} \sim L^z$ . We argue that the (non-trivial) scaling of the transient time in the OFC model is associated with the establishment of long-range spatial correlations in the steady state.

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## 1. Introduction

The idea of self-organized criticality (SOC) was introduced by Bak, Tang and Wiesenfeld (BTW) [1] as a possible paradigm for the widespread occurrence in nature of scale free phenomena. It refers to the intrinsic tendency of extended, non-equilibrium systems to spontaneously self-organize into a dynamical critical state. In general, SOC systems are driven externally at a very slow rate and relax with bursts of activity (avalanches) on a very fast (almost instantaneous) timescale. The standard signature of SOC is a power law distribution of avalanche sizes and in this sense the system is said to be critical. Typical physical realizations of this phenomena include, among others, earthquakes, forest fires and biological evolution (for a review, see, e.g., [2, 3]).

A number of simple lattice models have been developed to test the applicability of SOC to a variety of complex interacting dynamical systems [2, 3]. In general these models reach a stationary critical state after a sufficiently long transient time. However, not much attention has been paid to the self-organization process and studies have mainly concentrated on the

properties of different models at stationarity. This is partly justified by the fact that some of the most studied models, such as the BTW [1] and the Zhang [4] models, display a very simple transient time behaviour. Indeed in these models the relaxation time does not scale with system size (time is defined as the input energy per site in the system) [5]. On the other hand, there are also models with a more complex behaviour (see, e.g., [6]).

In this paper we investigate the approach to the stationary state in the so-called Olami–Feder–Christensen (OFC) model for earthquake dynamics [7]. This model has in recent years attracted a considerable amount of attention especially because it has been proposed as an example of a system displaying SOC behaviour even with a non-conservative dynamics [8–11]. One of the most important questions in this field is indeed whether a conserving local dynamics is a necessary condition for SOC [12, 13]. For example, it is well known that the BTW model is subcritical if dissipation is introduced [14]. The presence of criticality in the non-conservative OFC model is still debated [15, 16]. Recent numerical investigations, though, have shown that the model on a square lattice displays scaling behaviour, up to lattice sizes presently accessible by computer simulations [17, 18].

The present investigation complements previous analyses of the OFC model which were based on the study of the probability distribution for earthquake sizes. It provides further numerical support in favour of criticality in the non-conservative regime. Indeed, we will show that the model displays a non-trivial transient time behaviour: the relaxation time scales with system size and it is controlled by a dynamical critical exponent  $z$  that appears to be universal over a range of dissipation levels of the local dynamics. Moreover, we will establish the presence of long-range spatial correlations in the system. In doing so, we will be able to gain some insight into the mechanisms behind criticality in non-conserving systems, mechanisms that are very different from those at work in systems with a conservation law.

The plan of the paper is as follows. In section 2 we describe the model and briefly summarize previous findings relevant to our investigation. In section 3 we define the quantities of interest and present the results of our numerical study. Finally, in section 4 we discuss our main conclusions.

## 2. The model

The OFC model is a coupled map lattice model, where with each site  $(i, j)$  of a square lattice of linear size  $L$  is associated a real variable  $F_{ij}$ . In the initial state, at time  $t = 0$ , the values of the  $F_{ij}$  are chosen randomly in the uniform interval  $(0, F_c)$ . Subsequently the variables evolve according to the following two-step dynamical rules: (i) if all sites in the system are stable (i.e.  $F_{ij} < F_c$ ), they increase simultaneously and uniformly at a constant rate,

$$\frac{\partial F_{ij}(t)}{\partial t} = v \quad (1)$$

(ii) as soon as one of them reaches the threshold value  $F_c$ , the uniform driving is stopped and an ‘earthquake’ starts:

$$F_{ij} \geq F_c \Rightarrow \begin{cases} F_{ij} \rightarrow 0 \\ F_{nn} \rightarrow F_{nn} + \alpha F_{ij} \end{cases} \quad (2)$$

where ‘ $nn$ ’ denotes the set of nearest neighbour sites of  $(i, j)$  and  $\alpha$  is a parameter that controls the level of conservation of the dynamics ( $\alpha = 1/4$  corresponds to the conservative case). The ‘toppling’ rule (2) can possibly produce a chain reaction, which ends when there are no more unstable sites in the system. At that point, the uniform growth (1) starts again. In the following we will assume, without loss of generality, a unit growth rate, i.e.  $v = 1$ . A crucial point in the description of the model is the choice of boundary conditions and, in accordance

with previous investigations, we will consider open boundary conditions. These conditions imply that sites close to the boundaries topple according to (2) but have a smaller coordination number.

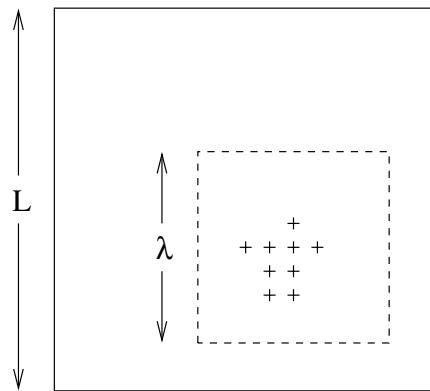
There is a clear separation of timescales in the system: earthquakes occur instantaneously on the slow timescale of the driving. The time in the system is therefore set by the slow time variable  $t$ . By construction, moreover, the time coincides with the input energy *per site* in the system. We will make use of this latter observation when we try to compare the behaviour of the OFC model with other models, such as the BTW model or the Zhang model.

After a sufficiently long transient time, the system settles into a stationary state, where the statistical properties of the model (e.g. the probability distribution for earthquake sizes) do not depend on time. In the BTW model and in the Zhang model, the transient time is relatively brief and does not scale with system size. On average, an input energy proportional to system size is needed to reach the steady state [5]. In contrast, transient times in the OFC model are known to be extremely long, especially for large lattices. For example, it was claimed in [8] that  $4 \times 10^8$  earthquakes are not enough to reach stationarity in a system of size  $L = 200$  for  $\alpha = 0.1$ . This conclusion was reached by observing the very slow convergence of the mean earthquake size to an asymptotic value during the transient time. However, in [8] a systematic investigation of the relaxation times for different  $\alpha$  and  $L$  was not attempted. A more quantitative approach to the problem was proposed in [10] by Middleton and Tang (MT). According to these authors a ‘self-organized’ region develops first close to the boundaries and propagates thereafter into the bulk of the system. The distance from the boundaries of the invasion front grows with time as a power law,  $d(t) \sim t^{\gamma(\alpha)}$ , with  $\gamma = 0.23 \pm 0.08, 0.63 \pm 0.08$  for  $\alpha = 0.07, 0.15$ , respectively. The system reaches stationarity when the SOC region crosses the whole sample ( $d(t) \sim L$ ). Assuming that the power law behaviour of  $d(t)$  holds until saturation, then the transient time of the system should scale as  $t_{\text{tr}} \sim L^{1/\gamma(\alpha)}$ . More recently, it has been suggested in [11] that two distinct relaxation times exist in the system, associated respectively with the power law region and the ‘tail’ (induced by finite size effects) of the distribution of earthquake sizes. According to this study, the former should stabilize much faster than the latter.

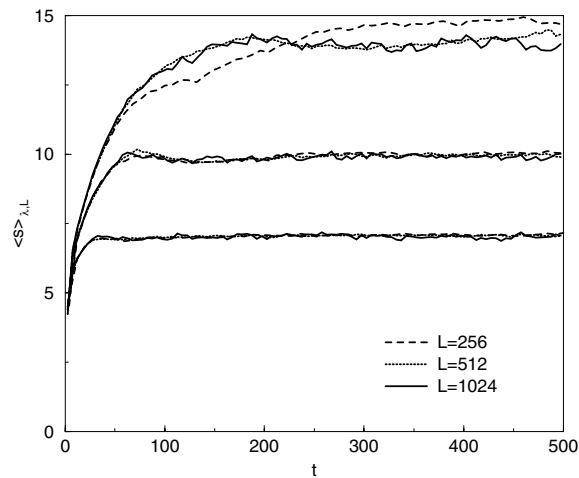
### 3. Results

Most of the studies on the OFC model at stationarity concentrate on the probability distribution of earthquake sizes,  $P_L(s)$ , where  $L$  is the size of the system and  $s$  is the total number of sites that topple during an earthquake. This probability distribution does not show simple finite size scaling, at least in the range of lattice sizes accessible to simulations at present [17]. In a recent paper [18], we have focused instead on the properties of earthquakes confined within a fictitious subsystem of linear size  $\lambda$  (see figure 1). The model is driven according to its usual dynamics but only those particular earthquakes that are entirely contained within the subsystem are counted. We have shown that if  $\lambda$  is sufficiently smaller than  $L$  the size distribution for this subset of earthquakes,  $P_{\text{conf}}(\lambda, s)$ , obeys ordinary finite size scaling, i.e.  $P_{\text{conf}}(\lambda, s) \simeq \lambda^{-\beta} f(s/\lambda^D)$ , where the exponents  $\beta = 3.6$  and  $D = 2$  are universal over a range of values of  $\alpha$ .

In this work we want to address the issues briefly summarized in the previous section concerning the approach to stationarity in the OFC model. In order to be able to formulate a scaling hypothesis and make use of collapse plots, we will proceed in a way similar to that of [18]. We will consider earthquakes localized within given subsystems, in particular subsystems placed (a) at the boundaries and (b) at the centre of the system. As it is a prohibitive task to determine the time evolution of the entire distribution  $P_{\text{conf}}(\lambda, s)$ , we will restrict ourselves



**Figure 1.** Schematic representation of earthquakes entirely confined within a subsystem of linear size  $\lambda$  (dashed line). Toppling sites are denoted with a cross.



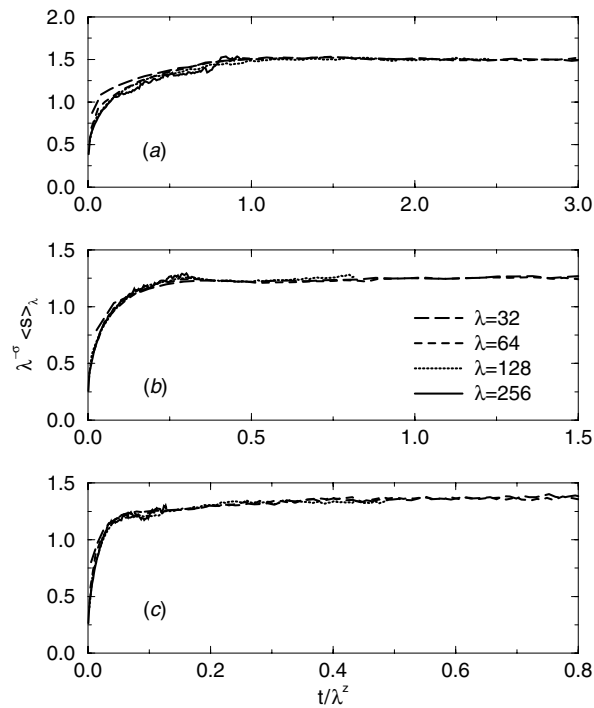
**Figure 2.** Average earthquake size  $\langle s \rangle_{\lambda,L}$  in a subsystem placed at the boundary as a function of time;  $\alpha = 0.18$  and, from bottom to top,  $\lambda = 32$ ,  $\lambda = 64$  and  $\lambda = 128$ .

to the mean earthquake size,  $\langle s \rangle_{\lambda,L}(t)$ , and, in general, to the  $q$ th moment (up to  $q = 4$ ) of the distribution,  $\langle s^q \rangle_{\lambda,L}(t)$ . To determine numerically these quantities we have run several simulations with different initial conditions, partitioning the time into bins of size  $\Delta t$ . Let  $n$  be the number of earthquakes occurring between time  $t - \Delta t/2$  and  $t + \Delta t/2$  in a given realization of the system and let  $s_1, \dots, s_n$  be the sizes of these earthquakes. Then we define

$$\langle s^q \rangle_{\lambda,L}(t) \simeq \frac{\langle [s_1^q + \dots + s_n^q]_{t-\Delta t/2}^{t+\Delta t/2} \rangle}{\langle [n]_{t-\Delta t/2}^{t+\Delta t/2} \rangle} \quad (3)$$

where  $\langle \dots \rangle$  denotes an average over different realizations of the system, i.e. over different initial conditions. For each system, the parameter  $\Delta t$  has been chosen small with respect to the transient time  $t_{tr}$  but large enough to collect reasonably accurate statistics.

We consider first the case of subsystems placed adjacent to a boundary of the system, in a symmetric position with respect to the corners. In figure 2, we report  $\langle s \rangle_{\lambda,L}$  as a function of time for  $\alpha = 0.18$  and some  $\lambda$  and  $L$ . We observe that if the linear dimension  $\lambda$  of the



**Figure 3.** Collapse plots of  $\langle s \rangle_\lambda(t)$  for a subsystem placed at the boundary and for (a)  $\alpha = 0.15$ , (b)  $\alpha = 0.18$  and (c)  $\alpha = 0.21$ . The value of the dynamical critical exponent is  $z = 1.3$ .

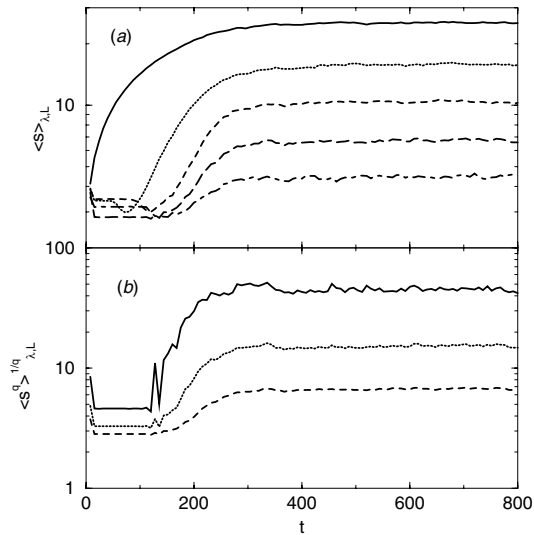
box is sufficiently smaller than the linear dimension of the system  $L$  (approximately  $L \geq 4\lambda$ ), then the curve  $\langle s \rangle_{\lambda,L}(t)$  becomes indistinguishable for different  $L$ . This has been verified also for other values of  $\alpha$  and  $\lambda$ . We will therefore denote with  $\langle s \rangle_\lambda(t)$  the mean earthquake size in this limit. It is already visible from figure 2 that the relaxation time of  $\langle s \rangle_\lambda(t)$  increases with  $\lambda$ . This is an indication in support of the scenario proposed by MT. Regions close to the boundaries reach stationarity sooner, signalling that an invasion front is moving towards the bulk of the system. Some of MT conclusions nonetheless will have to be modified as we will show later.

In order to describe quantitatively the invasion from the boundaries of the self-organized region we make the following simple scaling hypothesis:

$$\langle s \rangle_\lambda(t) = \lambda^\eta F(t/\lambda^z) \quad (4)$$

where  $\eta$  and  $z$  are two suitable critical exponents. In particular,  $z$  is a dynamical critical exponent that should satisfy  $z = 1/\gamma$ , where  $\gamma$  is the ‘invasion’ exponent as defined by MT. In the limit of  $t \rightarrow \infty$  the scaling function  $F(x)$  saturates to a constant, implying that the exponent  $\eta$  is related to the finite size exponents of the probability distribution  $P_{\text{conf}}(\lambda, s)$  by the relation  $\eta = 2D - \beta \approx 0.4$ . In figure 3, we report collapse plots of the form (4) for various values of  $\alpha$ . We observe that a reasonably good collapse could be obtained for all the  $\alpha$  if we choose the universal exponent  $z = 1.3 \pm 0.1$ . We are therefore led to conclude that a universal exponent  $z$  exists contrary to the claims by MT.

We consider next subsystems of different sizes placed at the centre of the system. We observe that the relaxation time does not depend on the size of the subsystems (see figure 4(a)). This confirms that the self-organization mechanism develops from the boundaries

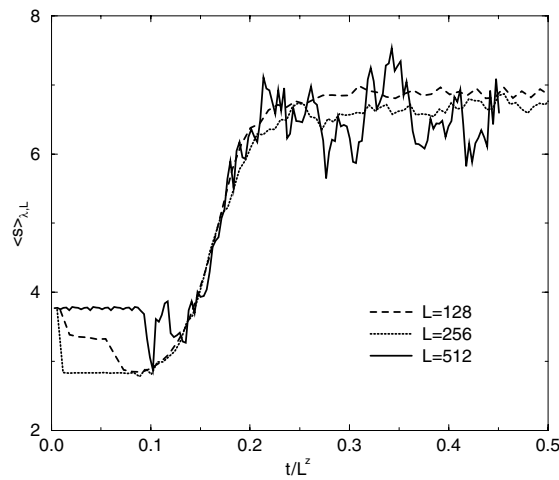


**Figure 4.** Time dependence of the average earthquake sizes in subsystems placed at the centre of the system for  $\alpha = 0.18$  and  $L = 256$ ; (a) average earthquake size  $\langle s \rangle_{\lambda,L}$  for, from bottom to top,  $\lambda = 16, 32, 64, 128, 256$  and (b)  $q$ th moment of the distribution for  $\lambda = 32$  and, from bottom to top,  $q = 1, 2, 4$ .

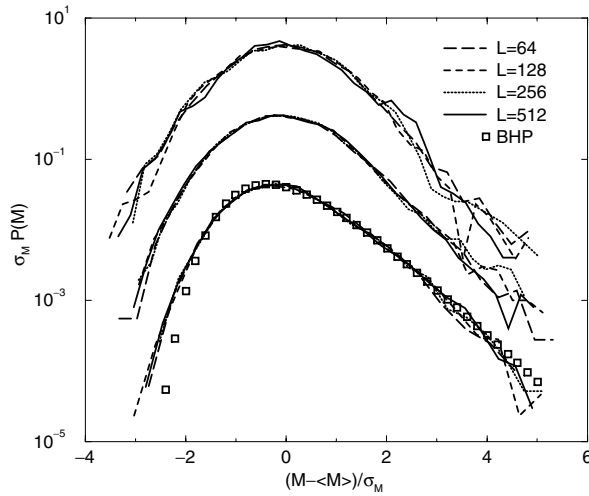
and that the system enters stationarity when the self-organized region spans the whole system. Only when the bulk of the system is reached by the self-organized region is stationarity settled so that the concentric subsystem of different sizes will inevitably reach stationarity at the same time. We have also tested the relaxation times for higher moments of the avalanche probability distribution to see whether different parts of the distribution (e.g. power law part and the ‘tail’) have different relaxation times as suggested in [11]. However, in our investigation we have not observed any significant difference in the relaxation times associated with different moments. We report as an example in figure 4(b) the behaviour of the first, second and fourth moments in a particular case.

The scaling equation (4) suggests that the transient time in the OFC model scales with system size as  $t_{\text{tr}} \sim L^z$ . One way to test this is by comparing the time behaviour of the average earthquake size in a central subsystem of size  $\lambda$  for different system sizes  $L$ . Indeed the asymptotic value  $\langle s \rangle_{\lambda,L}(t \rightarrow \infty)$  should not depend on  $L$ . We report as an example the case for  $\alpha = 0.18$  in figure 5, where we have rescaled the time by a factor  $L^z$ . The curves show some noisy behaviour, due to the difficulties in collecting good statistics (relatively few earthquakes occur in the bulk of the system as compared to the boundaries). Nonetheless the value deduced for the exponent,  $z \approx 1.3$ , is consistent with the determination made through the analysis of the earthquakes occurring at the boundaries. We have obtained similar results also for other values of  $\alpha$ . In addition, besides the average earthquake size, we have considered also the time behaviour of other quantities such as the roughness of the energy landscape (in analogy to surface growth problems) and the number of earthquakes per unit time. All these different quantities on average reach stationarity at the same time.

The algebraic divergence of the relaxation time with system size reflects the presence of long-range spatial correlations in the stationary state. Indeed if correlations were only short range, then one would expect that the transient time would not scale with system size. This is for example the case for the BTW model in  $d$  dimensions, where the height–height correlations



**Figure 5.** Collapse plots of  $\langle s \rangle_{\lambda,L}(t)$  for a subsystem of size  $\lambda = 32$  placed at the centre of a system of size  $L$ ; the conservation parameter is  $\alpha = 0.18$ . The value of the dynamical critical exponent is  $z = 1.3$ .



**Figure 6.** Rescaled probability distribution of the spatially averaged force in the system for, from bottom to top,  $\alpha = 0.21, 0.18, 0.15$ .  $\langle M \rangle$  and  $\sigma_M$  are, respectively, the average and the standard deviation of the distribution. The top and bottom curves have been shifted by a factor of 10, respectively, up and down for visual clarity. Squares represent the BHP (rescaled) probability distribution observed in an experiment on turbulence [20].

are algebraic but decays as fast as  $r^{-2d}$  ( $r$  being the distance between two sites) [19]. We have measured for various  $L$  and  $\alpha$  the probability distribution of the spatially averaged force in the system

$$M = \frac{1}{L^2} \sum_{i,j=1}^L F_{i,j}. \tag{5}$$



In a system with sufficiently short-range correlations, this probability distribution would tend, in the limit  $L \rightarrow \infty$ , to a Gaussian distribution around the mean due to the central limit theorem (this is indeed what results in the BTW model). Let  $\langle M \rangle$  and  $\sigma_M$  be, respectively, the average and the standard deviation of the distribution. In figure 6, we have plotted  $\log(\sigma_M P(M))$  versus  $(M - \langle M \rangle)/\sigma_M$  for various  $L$  and  $\alpha$ . Using these coordinates a Gaussian function would result in an inverse parabola. For each  $\alpha$  value the data of figure 6 collapse on a single function, which is clearly not Gaussian (deviations from Gaussianity are more pronounced for increasing  $\alpha$  values). This indicates that the central limit theorem does not hold in this case, not even for large  $L$ , suggesting that long-range algebraic correlations are present and therefore the sum (5) cannot be decomposed into a sum of independent terms. This observation is in agreement with the results reported in [13] where the presence of long-range spatial correlations were deduced from the behaviour of a suitably defined susceptibility,  $\chi \equiv (L\sigma_M)^2$ . It was claimed that  $\chi$  diverges as  $L^2$  and correspondingly that  $\sigma_M$  is, to leading order, independent of  $L$  (if  $M$  was a sum of uncorrelated variables,  $\sigma_M$  would decrease as  $1/L$ ). In our investigation we have found that  $\sigma_M$  slightly increases with  $L$  but asymptotically tends to a constant value, in accordance with [13] ( $M$  is a bounded variable so  $\sigma_M$  cannot grow indefinitely).

#### 4. Conclusions

In conclusion, in this paper we have examined the process of self-organization in the OFC model. By considering earthquakes confined within a given subsystem we have been able to clarify some of the issues related to this problem. In accordance with Middleton and Tang [10] we have found that SOC develops first close to the boundaries and subsequently invades the interior of the system. The invasion process is controlled by a dynamical critical exponent,  $z \simeq 1.3$ , which, contrary to previous claims, is universal over a range of values of the dissipation level of the local dynamics. We have shown, moreover, that the transient time in the system scales with system size as  $t_{tr} \sim L^z$ . This is a peculiarity of the OFC model as other ‘sandpile-like’ models (e.g. the BTW and the Zhang models) do not display any scaling in the transient time. We have associated this feature with the presence of long-range spatial correlations in the stationary state.

Our findings are in general agreement with recent works on the OFC model [17, 18]. Indeed we have provided complementary evidence (not based on the probability distribution for earthquake sizes) that the model is critical even in a non-conservative regime. Moreover it confirms that there is universality in the system and that finite-size scaling can be recovered by considering subsystems whose linear extent is sufficiently small.

Finally, it is interesting to remark that the probability distribution for the spatially averaged force in the system is somewhat reminiscent of a probability distribution observed in a confined turbulent flow experiment [20] (BHP). As a term of comparison we have reported in figure 6 the BHP functional form over-imposed on the curve for  $\alpha = 0.21$ . Attempts to link SOC systems to turbulent phenomena have long been suggested, but only recently has this been put on a firmer basis [21].

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